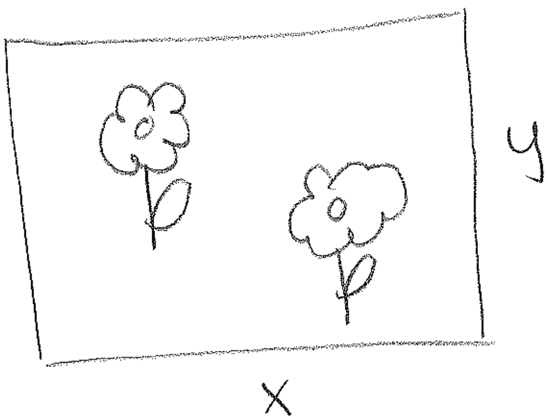
Feb. 18, 2014

Optimization

Suppose you want to fence off a garden, and you have 100m of fence. What is the largest area you can fence off?



Area: $A = xy$

want to maximize this.

But it's a function of two variables!

Use this

Restriction: $2x + 2y = 100$

$$2x + 2y = 100$$

$$2y = 100 - 2x$$

$$y = 50 - x$$

the substitute:

$$A = x(50 - x) = \underbrace{50x - x^2}_{\text{want to maximize this}}$$

Domain: $\underbrace{0 \le x \le 50}$

when it makes sense. If $x < 0$, can't have negative fencing

If $x > 50$, y is negative

$A = 50x - x^2$

We want the absolute maximum of $50x - x^2$ on the interval $[0, 50]$.

Derivative: $50 - 2x$

Critical Points: $50 - 2x = 0$ $\quad x = 25$

Test Critical Points and endpoints:

x	$A(x)$	
0	0	abs min
25	625	abs max
50	0	abs min

largest area you can fence off is $625\ m^2$

Solving an Optimization Problem:

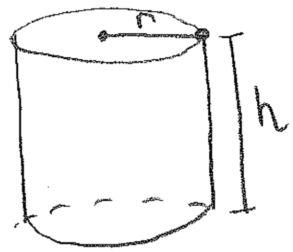
(1) Draw a picture

(2) Introduce Notation

(3) Express quantity to be maximized (or minimized) in terms of one variable

(4) Use derivatives to find absolute max (or min)

Ex: We want to manufacture a cylindrical can to hold 1 L of oil. Find the dimensions that will use the least amount of metal.

(1) 

Note: $1\,L = 1000\ cm^3$

(2) $V_{cyl} = \pi r^2 h = 1000$ S = surface area

(3) $S = \underbrace{2\pi r^2}_{\text{top/bottom}} + \underbrace{2\pi r h}_{\text{sides}}$ need to express in terms of one variable.

Note $1000 = \pi r^2 h$ so $h = \frac{1000}{\pi r^2}$

thus $S = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right)$

$S = 2\pi r^2 + \frac{2000}{r}$

(4) $\frac{dS}{dr} = 4\pi r + \frac{-2000}{r^2}$

Critical points: derivative doesn't exist when $r=0$

set $0 = 4\pi r + -\frac{2000}{r^2}$

$0 = 4\pi r^3 - 2000$

$\frac{2000}{4\pi} = r^3$

$\underbrace{\left(\frac{2000}{4\pi}\right)^{1/3} = 5.419}_{\text{minimum}}$

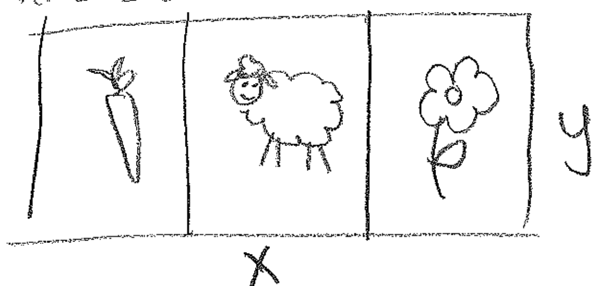
Notice $r > 0$

− +

5.419

plug into derivative

Ex: We have 100 m of fence, and we want to set up fences to divide land into 3 equal parts like so:



What is the largest area you can enclose?

area $= xy$ Fence used: $2x+4y=100$

$$x+2y=50$$
$$y=\frac{50-x}{2}$$

$$\text{area} = x\cdot\left(\frac{50-x}{2}\right)$$
$$= \frac{1}{2}(50x-x^2) = f(x)$$

$$f'(x) = \tfrac{1}{2}(50-2x)$$
$f'(x) = 0 = \tfrac{1}{2}(50-2x)$ when $x=25$

Note: $0<x<25$

at $x=25$, area $= 25\cdot\left(\frac{50-25}{2}\right)$
$$= \underbrace{\frac{25^2}{2}}_{\text{max or min?}}$$

Second Derivative test: check concavity.

$f''(x) = -1$

Concave down everywhere
$\Longrightarrow$ 25 is a max

So $\frac{25^2}{2}$ is max area.